# Character of a Representation

$\varphi : G \to GL(V)$ representation.

The character of $\varphi$ is a function

$\chi_\varphi : G \to \mathbb{C}$ defined by

$$\chi_\varphi(g) := \underline{\mathrm{Tr}}(\varphi_g) \qquad \varphi_g : V \to V$$

If $\varphi : G \to GL_n(\mathbb{C})$, $\quad \varphi_g = (\varphi_{ij}(g))_{1 \le i,j \le n}$

then $\chi_\varphi(g) = \sum_{i=1}^{n} \varphi_{ii}(g)$.

**Prop**: If $\varphi \sim \psi$, then $\chi_\varphi = \chi_\psi$

**Proof**: $T \in \mathrm{Hom}_G(\varphi, \psi)$, isomorphism

$$\psi_g = T \varphi_g T^{-1} \Rightarrow \mathrm{Tr}(\psi_g) = \mathrm{Tr}(\varphi_g)$$

**Example:** If $\varphi: G \to GL_1(\mathbb{C}) = \mathbb{C}^\times \subseteq \mathbb{C}$

then $\chi_\varphi(g) = \varphi(g)$

[write "$\chi_\varphi = \varphi$" if $\dim \varphi = 1$]

**Prop:** $\chi = \chi_\varphi$, $\varphi: G \to GL(V)$

$\chi(e) = \dim V$, $\chi(g^{-1}) = \overline{\chi(g)}$

and $\chi(g) =$ finite sum of $\zeta^k$ some $k$s

where $\zeta = e^{2\pi i/n}$,

$n = o(g)$

**Proof:** $\chi(e) = \operatorname{Tr}(\varphi_e) = \operatorname{Tr}(I) = \dim V$

Assume $\varphi: G \to GL_n(\mathbb{C})$, $o(g) = d$.

$A := \varphi_g$ so $A^d = I$

so $A \sim \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, $\lambda_k^d = 1$

so $\lambda_k = \zeta^{m_k}$, $\zeta = e^{2\pi i/d}$, some $m_k$

$\chi(g) = \zeta^{m_1} + \dots + \zeta^{m_n}$,

$$A^{-1} = \varphi_{g^{-1}} \sim \mathrm{diag}(\lambda_1^{-1}, \dots, \lambda_n^{-1})$$

$$\mathrm{diag}(\overline{\zeta^{m_1}}, \dots, \overline{\zeta^{m_n}})$$

$$\Rightarrow \quad \chi(g^{-1}) = \overline{\zeta}^{m_1} + \dots + \overline{\zeta}^{m_n} = \overline{\chi(g)}$$

**Prop:** If $\varphi = \varphi^{(1)} \oplus \dots \oplus \varphi^{(r)}$

then $\chi_\varphi = \chi_{\varphi^{(1)}} + \dots + \chi_{\varphi^{(r)}}$

**Prop:** $\varphi : G \to GL(V)$, $\theta : G \to GL_1(\mathbb{C}) = \mathbb{C}^\times$

Then $\chi_{\theta\varphi} = \theta \cdot \chi_\varphi$